ON THE EQUATIONS OF MOTION OF A RAREFIED GAS

(OB URAVNENIIAKH DVIZHENIIA RAZREZHENNOGO GAZA)

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1. Introduction. Strictly speaking, the Navier-Stokes equations of aerodynamics are legitimately applicable to flows in which the characteristic length L is much larger than the length of the free path of the molecules λ.

For larger values of λ (i.e. for less dense gases) or smaller values of L (i.e. for flows with large gradients), the motion of the gas cannot, generally speaking, be treated as the motion of a continuous medium - account must be taken of its molecular structure. The motion of a monatomic gas (in which triple collisions of molecules are extremely rare, and the mutual interaction of the molecules arises only during collisions between molecules, occupying intervals of time which are short by comparison with the intervening time between collisions) is assumed in the kinetic theory of gases to be characterized by a distribution function $f(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) = f(x_i, \xi_i)$.

The distribution function is normalized in such a way that the expression $f(x_i, \xi_i)\, d\xi_1\, d\xi_2\, d\xi_3 = f dq$ represents the number of molecules per unit volume at the point x_i, moving with velocities between ξ_i and $\xi_i + d\xi_i$. The hydrodynamic quantities are expressed in terms of the distribution function as follows:

$$\rho(x_i, t) = mn = m\iiint f(x_i, t, \xi_i)\, d\xi_1\, d\xi_2\, d\xi_3 = m\int f d\mathbf{q} \qquad \text{(density)} \qquad (1.1)$$

$$u_i(x_i, t) = \frac{1}{n}\int f\xi_i d\mathbf{q} \qquad \text{(mean velocity of the molecules)} \qquad (1.2)$$

$$\frac{3}{2}kT(x_i, t) = \frac{m}{2n}\int f c^2 d\mathbf{q} \qquad \text{(temperature)} \qquad (1.3)$$

$$p = knT \qquad \text{(pressure)} \qquad (1.4)$$

Here k is the Boltzmann constant, m is the mass of the molecule, n is the number of molecules per unit volume and $c^2 = c_1{}^2 + c_2{}^2 + c_3{}^2$, where $c_i = \xi_i - u_i$ is the velocity of the molecule with respect to coordinates moving with the mean velocity of the molecules u_i. Expressions of the type

$$\int f\,(x_{i1}t_1\xi_i)\, c_i^{\alpha} c_j^{\beta} c_k^{\gamma} d\mathbf{q}$$

are called moments. Moments with a simple physical significance are:

$$P_{ij} = m\int f c_i c_j d\mathbf{q} = p_{ij} + p\delta_{ij} \qquad \text{(stress tensor)} \qquad (1.5)$$

$$\delta_{ij} \qquad \text{(Kronecker delta)}$$

$$q_i = \frac{m}{2}\int f c^2 c_i d\mathbf{q} \qquad \text{(vector current of energy)} \qquad (1.6)$$

In the absence of body forces, the distribution function satisfies Boltzmann's equation:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \xi\frac{\partial f}{\partial t_i} = 2\pi\iiiint (f^{*\prime}f' - f^{*}f)\, gb\, db\, d\mathbf{q}^{*} = \Phi \qquad (1.7)$$

The right-hand side of Boltzmann's equation gives a measure of the number of molecules with velocities in the interval $[\xi_i, \xi_i + d\xi_i]$ as a result of collisions with molecules having velocities in the interval $[\xi_i^*, \xi_i^* + d\xi_i^*]$.

In (1.7) $g = |\mathbf{q} - \mathbf{q}^*|$ is the modulus of the relative velocity of the colliding molecules, b is the approach distance. The integration is taken over all possible velocities $\mathbf{q}^*$ of the incident molecules and with approach distances from $b = 0$ up to a certain $b_{\max}$, such that when $b > b_{\max}$ it can be assumed that the interaction potential of the molecules ϕ is equal to zero . The primes after f and f^* in the right hand side of (1.7)*show that these functions are evaluated for velocities ξ_i' and $\xi_i^{*\prime}$ respectively, of the molecules after the collision. These velocities are determined from the velocities of the molecules before the collision, their approach distance and the interaction law of the molecules. The structure of the right-hand side of the equation even for the simplest laws of interaction of molecules turns out to be extremely complex, which represents one of the fundamental difficulties of the application of equation (1.7).

If equation (1.7) be multiplied by m, mc_i or $1/2\ mc^2$ ** and integrated with respect to $\mathbf{q}$, then the integral on the right-hand side is identically zero [1, 2], and from the left-hand side we obtain, respectively,

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*It is assumed that the potential possesses spherical symmetry.

**I.e. by the so-called summatory invariants.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \qquad \text{(equation of conservation of mass)} \qquad (1.8)$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial P_{ij}}{\partial x_j} = 0 \qquad \text{(equation of motion)} \qquad (1.9)$$

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i}(u_i p) + \frac{2}{3} P_{ij}\frac{\partial u_i}{\partial x} + \frac{1}{3}\frac{\partial q_i}{\partial x_i} = 0 \quad \text{(equation of heat flow)} \quad (1.10)$$

Continuing this process further (i.e. multiplying equation (1.7) by $c_i^{\alpha} c_j^{\beta} c_k^{\gamma}$ and integrating), we obtain equations for P_{ij}, q_i and higher moments *. The right-hand sides of the equations, however, do not now vanish identically, and are expressed in the general case in terms of the moments of all orders. The conjoint infinite system of equations, when obtained, is equivalent to Boltzmann's equation (1.7).

This system, however, is so complex that until now only very degenerate exact solutions have been obtained by this method.

The equations (1.8) - (1.10) determine all the hydrodynamic quantities ρ, u_i and p. In order that this system should be self-contained, it is necessary that the stress tensor P_{ij} and the vector flow of heat q_i be expressed in terms of the hydrodynamic quantities and their derivatives. It is easy to show, however, that this is not possible for every value of the Knudsen number $K = \lambda/L$.

In fact, suppose that there are two planes with temperatures T_1 and T_2 (for definiteness $T_1 > T_2$), separated by distance L from one another. If the length of the free path $\lambda >> L$, then the temperature between the planes $T(T_2 < T < T_1)$ is uniform. Nevertheless, heat will be transferred from plane 1 to plane 2, i.e. in this case c_i at a certain internal point cannot be expressed in terms of the hydrodynamic quantities at this point and of their derivatives of any order. On the other hand, when $\lambda << L$, the Navier-Stokes equations, which are obtained from (1.8) - (1.10), are applicable, and it is well known that P_{ij} and q_i are expressible in terms of the derivatives of the hydrodynamic quantities. Accordingly, it may be thought that if in general it is possible to represent P_{ij} and q_i in the form of functions of the hydrodynamic quantities and their derivatives, then this can be done only for a certain range of slightly rarefied gases.

For small Knudsen numbers Enskog [1] sought the distribution function in the form of a series:

$$f = f_0 + Kf_1 + K^2 f_2 + \dots \qquad (1.11)$$

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* Different forms of the corresponding infinite systems of equations have been obtained by Grad [2] and Truesdell [3].

The right-hand side of equation (1.7) has order K^{-1}. Accordingly, substituting the series for f in (1.7) and equating terms of the same order in K, we obtain a recurrent system of integral equations for the determination of f_0, f_1. The solution of the equation for f_0 is Maxwell's equilibrium function

$$f_0 = n\left(\frac{m}{2k\pi T}\right)^{3/2}\exp\left(\frac{m}{2kT}c^2\right)$$

If we limit ourselves to this approximation, then $P_{ij} = p\delta_{ij}$ and $q_i = 0$ and the equations (1.8) - (1.10) become Euler's equations.

Substituting the f_0 found in this way in the integral equation for f_1 and solving it, we find f_1. Restricting consideration to two terms of the series, we obtain the Navier-Stokes equations. In succeeding approximations we obtain Burnett's equation [1, 4], and so on. The solution of the resulting integral equations, however, is extremely complicated and already in Burnett's approximation we are reduced to seeking a solution for the distribution function in a particular form.

Grad [2] found these same approximations by a somewhat different method for Maxwellian molecules, applying the method of iteration to a simplified system of equations for the moments. In order to achieve simplification he had to restrict himself to a study of the equations, corresponding to a distribution function of a particular form, which compels one to doubt whether all the terms in the equations were included.

In the present paper Boltzmann's equation is simplified in such a way that it is not only easy to find from it the distribution function to an arbitrary degree of approximation, but also it is easy to find the remainder of the series, which makes it possible in any practical case to estimate the error of the accepted approximation for the flow of a slightly rarefied gas.

The approximate form of Boltzmann's equation obtained in this paper makes possible in this case an important simplification of the calculations.

2. Simplification of Boltzmann's equation. The right-hand side of equation (1.7) can be written in the form of the difference:

$$\Phi = \Phi_1 - \Phi_2 \tag{2.1}$$

in which Φ_2 is defined as the number of molecules having velocities in the interval $[\xi_i, \xi_i + d\xi_i]$ and leaving it as a result of collisions, whilst Φ_1 is the number of molecules acquiring this velocity as a result of collisions.

Let us consider any one of the molecules, possessing the velocity ξ_i. The number of such molecules at a given moment of time t per unit volume at the point x_i is equal to $f(x_i, t, \xi_i) = f$. Let us consider the collisions of this molecule with molecules, possessing velocities ξ_i^* [the number of such molecules per init volume is equal to $f(x_i,\ t, \xi_i^*)$]. Let us assume Maxwell's law for the interaction of molecules, i.e. that the force of molecular interaction $F = ar^{-5}$, where r is the distance between the molecules and a is a constant. In other words, the molecule 2 with velocity ξ_i interacts with molecules 1 with velocities ξ_i^* for any approach distance b. However, the greater the approach distance, the less the interacting molecules change each other's motion. Accordingly, from each group of interacting molecules it is expedient to select only those which undergo a large change in their momentum and energy in the collision. Let us consider the collision of the molecules (Fig.1) in coordinates moving with the velocity ξ_i of the molecule 2 before the collision. Let us direct the axis of y along the direction of relative motion g. The components of velocity of the molecule 1 after the collision are given by

$$V_{1x} = \frac{1}{2} g \sin \alpha, \quad V_{1y} = \frac{1}{2} g \cos \alpha + \frac{1}{2} g$$

where α is the deflection of molecule 1. Let b be such that the deflection α is small. Then the change of momentum is of order $mg\alpha$. On the other hand, the impulse received by molecule 1 with the Maxwellian interaction law is (Fig.1)

$$\int F \cos \theta \, dt = \int_{-\infty}^{+\infty} F \cos \theta \, \frac{dy}{g} = \int_{-1/2\pi}^{1/2\pi} \frac{a}{b^4 g} \cos^5 \theta d\theta = \frac{\text{const}}{b^4 g}$$

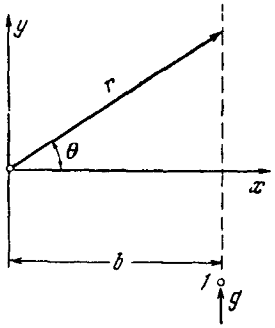

Fig. 1.

Accordingly the ratio of the change in momentum to the characteristic change of momentum mg is

$$\frac{mg\alpha}{mg} \sim \frac{\text{const}}{b^4 g^2}$$

If we require that, when b is equal to b^0 - the effective radius of a collision - this ratio should equal a certain small quantity which is the same for all collisions (i.e. all g), then the effective radius of the collision b^0 must be related to g by the formula b^0 = const. $g^{-1/2}$. Then the effective collision cross-section σ^0 is equal to

$$\sigma^0 = A/g \tag{2.2}$$

where A is a constant depending on the sort of molecule. This same relationship is obtained also from consideration of the change of energy.

Accordingly, of all collisions of the molecule 2 under consideration with molecules 1, we shall consider only those which are included within a cylinder of height g, and having a base area of $\sigma^0 = A/g$. The number of such molecules is obviously equal to Af^*. Then the total number of collisions of the molecule considered, with molecules having velocities ξ_i, is

$$\Phi_2 = Af\int f^* d\mathbf{q} = Anf \tag{2.3}$$

Accordingly, a part of the collision integral Φ_2 is expressed in a simple form. In order to simplify the very complex part Φ_1 of the collision integral, we assume that the colliding molecules acquire as a result of the collision the most probable distribution for a given number of molecules, and given momentum and energy. According to (2.3), the number of colliding molecules is

$$\int Anf d\mathbf{q} = An^2 \tag{2.4}$$

Their momentum and mean kinetic energy (temperature) are, respectively,

$$m\int Anf\xi_i d\mathbf{q} = Amu_i n^2, \qquad \frac{m}{2}\int Anfc^2 d\mathbf{q} = \frac{3}{2}kTn^2A \tag{2.5}$$

Then it is easily seen that the most probable distribution is the Maxwell Distribution

$$\Phi_1 = An^2\left(\frac{m}{2k\pi T}\right)^{3/2}\exp\left(-\frac{mc^2}{2kT}\right) = Anf_0 \tag{2.6}$$

where f_0 is Maxwell's distribution function, corresponding to the given function f. Accordingly, the distribution function of the colliding molecules in our case is proportional to the Maxwell distribution corresponding to the parameters of the whole stream.

The assumption concerning the most probable distribution of the molecules after collision is justified by the following arguments. It is well known (see [2], and also Section 3 of the present paper) that, whatever the distribution of the molecules, it tends to the equilibrium (most probable) distribution according to an exponential law. Moreover, the relaxation time is of the order of the time between collisions, i.e. when the molecules collide once, the distribution of the molecules approximates appreciably to equilibrium.

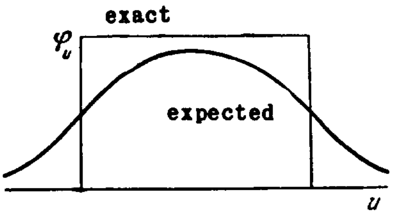

Fig. 2.

In Fig. 2, 3 and 4 are shown examples of the tendency to the equilibrium distribution for three degenerate simple cases, very far from equilibrium before collision; all the cases were calculated for rigid spheres. In each of the cases, two groups of spheres collide, one of the groups being at rest before the collisions. The other group is moving with constant velocity in the case depicted in Fig. 2. In this Fig. is shown the distribution of molecules with respect to velocity in the direction of the initial motion of the molecules of the second group. In Fig. 3 is depicted the result of the collision of the molecules for the case when the second group consists of two equal parts, moving towards one another with the distribution $B|u|\,e^{-k|u|}$ where B and k are constants. In the third case (Fig.4), the second group of molecules moves with spherical symmetry towards the centre, with distribution of the molecules according to their velocities Bc^2e^{-kc}.

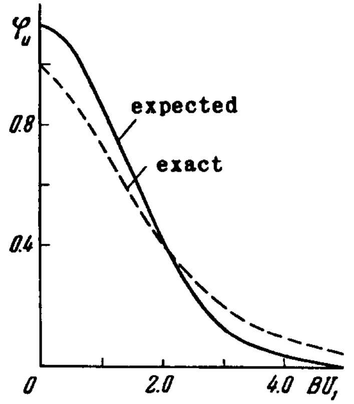

Fig. 3.

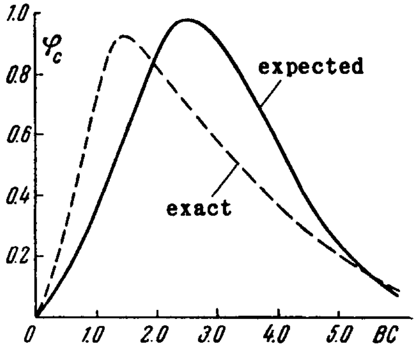

Fig. 4.

All these examples show that, after one collision, the distribution of velocities has come very close to equilibrium. Finally, it is possible to contrive special examples when this is not so. Such cases, however, must occur extremely rarely.

Accordingly, Boltzmann's equation (1.7) can be rewritten in the following simple approximate form:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} = An\,(f_0 - f) \tag{2.7}$$

The right-hand side of Boltzmann's equation, in a form analogous to (2.7), has been used by many authors on the basis that, for a state close to equilibrium, the trend of the velocities towards equilibrium is proportional to the deviation from the equilibrium state. In the paper [5] an analogous equation was applied without justification to all Knudsen numbers. The analysis presented above shows under what assumptions Boltzmann's equation can be represented in the form (2.7). In particular, it follows from the analysis that equation (2.7) is unsuitable for Coulomb interactions, to which it is applied in the paper [5].

3. The equations of motion of slightly rarefied gases. Assuming that the hydrodynamic quantities ρ, u_i and T appearing in f_0 are given, we can write the solution of equation (2.7) in the form

$$f(x_{i0} + \xi_i t, t, \xi_i) = \frac{1}{M(t)} \left\{ A \int_0^t n f_0 M(t)\, dt + f(x_{i0}, O, \xi_i) \right\}, \qquad M(t) = \exp \int_0^t An\, dt \tag{3.1}$$

Integrating successively by parts, we obtain

$$f = \sum_{k=0}^{N} a_k (x_{i0} + \xi_i t,\ t, \xi_i) + \tag{3.2}$$

$$+ \frac{1}{M(t)} \left[f(x_{i0},\ O,\ \xi_i) - \sum_{k=0}^{N} a_k (x_{i0},\ O,\ \xi_i) \right] + \frac{1}{M(t)} \int_0^t \frac{da_N}{dt} M(t)\, dt$$

$$\left(a_k = \frac{(-1)}{An} \frac{da_{k-1}}{dt}, \qquad a_0 = f_0 \right)$$

If the series converge for all points of the region as $N \to \infty$, then (3.2) transforms to Enskog's series (1.11). It is interesting to notice that in this case the distribution function (and consequently all moments as well) at a certain point of the flow is completely determined by the hydrodynamic quantities and their derivatives, and does not depend directly upon the boundary conditions.

If, however, there is a point $(x_i,\ O,\ \xi_i)$ in the region, at which Enskog's series does not converge (for example, inside a sharp density discontinuity), then the boundary conditions are not completely eliminated. However, at distances of several free paths of the molecules from this region (for example, from the discontinuity) the factor M^{-1} in (3.2) becomes so small that at this point it is possible to use a finite portion of the series, containing only the parameters relating to the given point.

The formula (3.2) makes it possible not merely to write out all the terms of Enskog's series, but also gives an expression for the remainder of the series, which in any practical case enables one to estimate the error of the accepted approximation. From the form of the remainder term it is obvious that the series converges asymptotically for large An (i.e. for slightly rarefied gases).

Substituting the distribution function (3.2) in (1.5) and (1.6), let us find P_{ij} and q_i expressed in terms of the hydrodynamic quantities and their derivatives.

Thus, retaining only two terms (the Navier-Stokes approximation), we obtain

$$P_{ij} = p\delta_{ij} - \frac{kT}{A}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{2}{3}\frac{kT}{A}\frac{\partial u_k}{\partial x_k}\delta_{ij} \qquad (3.3)$$

$$q_i = -\frac{5}{2}\frac{k^2T}{Am}\frac{\partial T}{\partial x_i}$$

Accordingly, for co-efficients of viscosity μ and heat conduction λ we obtain

$$\mu = \frac{kT}{A}, \qquad \lambda = \frac{5}{2}\frac{k^2T}{Am} = \frac{5}{2}\frac{k}{m}\mu \qquad (3.4)$$

The exact Boltzmann equation also leads to a linear dependence of λ and μ upon the temperature for Maxwellian molecules. The coefficient linking λ and μ in the exact theory, however, is 15/4 and not 5/2.

Let us consider a certain state of the molecules, when $u_i = 0$, $n =$ const. and $T =$ const. in the whole of space, whilst P_{ij}, q_i and the other moments are not equal to zero,

Then the distribution function satisfies the condition

$$\frac{df(t)}{dt} = An\,(f_0 - f) \qquad (3.5)$$

where n and f_0 are constants with respect to time and space. The solution of this equation, clearly, is the expression

$$f(t) = f_0(t) + e^{-Ant}\,[f(0) - f_0(0)] \qquad (3.6)$$

Then for the moments we have

$$p_{ij}(t) = e^{-Ant}p_{ij}(0), \qquad q_i(t) = e^{-Ant}q_i(0) \text{ and so on,}$$

or, eliminating A by means of (3.4), we obtain

$$p_{ij}(t) = e^{-t/\tau} p_{ij}(0), \qquad q_i(t) = e^{-t/\tau} q_i(0) \text{ and so on,} \tag{3.7}$$

where $\tau = \mu/p$ is the relaxation time.

The exact solution for this case is, however, [2]

$$p_{ij}(t) = e^{-t/\tau} p_{ij}(0), \qquad q_i(t) = e^{-2/3\, t/\tau} q_i(0) \text{ and so on,}$$

Here again, the exact and the approximate solutions differ in the constants.

Finally, let us introduce for comparison the infinite system of equations for the moments, equivalent to Boltzmann's equation, in the case of Maxwellian molecules [2] and the parallel system (on the right), obtained from (2.7), i.e.

$$\begin{array}{ll}
\dfrac{\partial p_{ij}}{\partial t} + \cdots = -Anp_{ij}, & \dfrac{\partial p_{ij}}{\partial t} + \cdots = -Anp_{ij} \\
\dfrac{\partial q_i}{\partial t} + \cdots = -\dfrac{2}{3} Anq_i, & \dfrac{\partial q_i}{\partial t} + \cdots = -Anq_i \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots
\end{array}$$

the exact and the approximate systems differ only in the coefficients.

It would be possible to derive Burnett's equation and higher approximations from the series (3.2). The complexity, however, even of Burnett's approximation leaves little hope of the possibility of making use of them.

Equation (2.7) and formula (3.1) are suitably employed for assessing the remainder term of Enskog's series also in those cases when an explicit form for the distribution function itself is required for analysis.

4. The motion of a highly rarefied gas. For large Knudsen numbers ($\lambda/L \gg 1$) the influence of collisions is small and in the first approximation can be neglected. Then Boltzmann's equation has the solution:

$$f^{(0)}(x_{i0} + \xi_i t, t, \xi_i) = f(x_{i0}, O, \xi_i) \tag{4.1}$$

Accordingly, for given initial and boundary conditions it is easy to find the distribution of the velocities of the molecules at any point of the region, and consequently, in the first approximation $n^{(0)}$, $T^{(0)}$, $u_i^{(0)}$ and $f_0^{(0)}$. Substituting the $f^{(0)}$ and $f_0^{(0)}$ so obtained in the right hand side of the approximate Boltzmann equation, we obtain equations for the determination of $f^{(1)}$ and so on.

In so far as the collisions play a comparatively minor role for high degrees of rarefaction, then it is possible to hope that the assumptions made in the simplification of Boltzmann's equation are completely admissible for calculations of flows approximating to free-molecule flow.

Generally in those cases when the approximate Boltzmann equation cannot guarantee the necessary accuracy, it should be regarded as a mathematical model of the exact Boltzmann equation on which, thanks to the comparative simplicity of the approximate equation, one can study the behavior of the distribution function.

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